DUALITY AND INFERENTIAL SEMANTICS

Abstract. It is well known that classical inferentialist semantics runs into problems regarding abnormal valuations [? ? ?]. It is equally well known that the issues can be resolved if we construct the inference relation in a multiple-conclusion sequent calculus. The latter has been prominently developed in recent work by Greg Restall [?], with the guiding interpretation that the valid sequent $\Gamma \vdash_L \Delta$ says that the simultaneous assertion of all of Γ with the denial of all of Δ is incoherent. However, such structures face significant interpretive challenges [? ? ?], and they do not provide an adequate grasp on the machinery of the duality of assertions and denials that could (a) provide an abstract account of inferential semantics; (b) show why the dual treatment is semantically superior. This paper explores a slightly different tack by considering a dual-calculus framework consisting of two, single-conclusion, inference relations dealing with the preservation of assertion and the preservation of denial, respectively. In this context, I develop an abstract inferentialist semantics, before going on to show that the framework is equivalent to Restall's, whilst providing a better grasp on the underlying proof-structure.

1. Introduction

1.1. **Weaknesses of inferential semantics.** Typically, model-theoretic consequence is thought to delineate the correctness of inference because it is (in some sense) reducible to the categorical notion of truth-preservation. In rough:

> β is derivable from $\alpha_1, \alpha_2 ... \alpha_n$ whenever β is a logical consequence of $\alpha_1, \alpha_2 ... \alpha_n$..

In the Bolzano-Tarski tradition, this is typically thought of as follows. If Γ is a theory of $\mathscr{L}$, and V is a semantic structure (typically thought of as a model) for $\mathscr{L}$, then V is a model of Γ: $V \models \Gamma$ if, for every $\alpha \in \Gamma$, $V \models \alpha$. And, α is a logical consequence of Γ ($\Gamma \models \alpha$) if, for every model V of Γ, $V \models \alpha$. Inferential derivability, on the other hand, is usually understood as derivability in a formal system (sequent calculus; natural deduction; Hilbert system), where $\Gamma \vdash_L B$ is valid in a formal system L, whenever B can be derived from Γ by means of the axioms and inference rules of L.

Then, soundness and completeness ensure that the proof-theory is both correct (according to the semantics), and that it is also strong enough to prove any valid argument. The inference structure gives us epistemic access to validity in the form of proofs, derivable sequents, and so on. But, so the thought goes, we require the semantic structure to tell us where the counter-models are, lest we attempt an invalid proof. So-called paradoxical connectives such as "tonk" [?] provide grist to the mill for this order of priority. Given this, there is a residual feeling that we are not able to provide an account of "real" validity limited to inferential structures alone.[1] In relation to classical logic at least, this feeling is well-supported by the issue that, in standard single-consequence classical inference structures (such as natural deduction), the rules defining $\vee$ and $\neg$ fail to characterise the classical truth-functions [? ?]. Take $\vee$ as example, with the rule $R^{\vee}$ as follows:

$$\frac{\alpha}{\alpha \vee \beta}\vee I^{\alpha} \qquad\qquad \frac{\beta}{\alpha \vee \beta}\vee I^{\beta}$$

$$\frac{\alpha \vee \beta \qquad \begin{matrix}[\alpha^u]\\ \sigma\end{matrix} \qquad \begin{matrix}[\beta^v]\\ \sigma\end{matrix}}{\sigma}\vee E^{u,v}$$

Consider this rule in relation to the generalised notion of a semantic structure given below.

Definition 1. (Semantic structure) Define a semantic structure as an ordered pair $\langle S, V\rangle$, where S is the enumerable set of formulas in a language $\mathscr{L}$, and V is a valuation space representing a structure of admissible valuations on the language.[2] Then, let, $\mathcal{V} = \{1, 0\}$ be a set of truth-values. A valuation v is a function on $\mathscr{L}$ assigning a truth-value $\in \mathcal{V}$ to a formula $\alpha \in S$, where $v : S \to \{\mathcal{V}\}$.

Semantically, it is usual to say that that the classical semantic structure V_{CPL} is recursively induced by the truth-conditional interpretation of each piece of logical vocabulary. In other words, V_{CPL} contains those valuations that obey the classical truth-conditional clauses for the connectives, which, in most cases, is equivalent to taking those valuations recursively induced by the defined truth-functions for the connectives. However, from an inferentialist point of view, we might look for the set of valuations which are consistent with a set of inferential rules such as $R^{\vee}$. In rough, by consistent, I mean any valuation v which satisfies an inference $\Gamma \mathrel{|\!\!\frac{}{\;L\;}} \alpha$ (in an inference structure L) so that $v(\alpha) = 1$ whenever $v(\Gamma) = 1$, and that v is consistent with a rule R iff v satisfies the conclusion of R whenever v satisfies the premises.

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[1] Stephen Read [?] sums up this view:

> What is good about the notion of proof-theoretic validity is that it recognises that what rules one adopts determines the meaning of the logical terms involved and commits one to accepting certain inferences as valid. What is bad is to infer from this that those inferences really are valid. Proof-theoretic validity serves an epistemological function to reveal how those inferences result from the meaning-determining rules alone. But it cannot serve the metaphysical function of actually making those inferences valid. Validity is truth-preservation, and proof must respect that fact.

[2] See [? ?].

The issue for $R^{\vee}$ arises for cases in which $v(\alpha) = v(\beta) = 0$ for $E^{u,v}$. In this case, we can not ensure that $v(\alpha \vee \beta) = 0$ since we are able only to conditionally infer σ from $\alpha \vee \beta$, given independent sub-derivations to σ (which will not figure in the immediate sub-formulas of complex formulas involving $\vee$). We do not have in schematic form all of the relevant information encoded within the immediate sub-formulas involved in the derivation, so we can not construct, from $E^{u,v}$, a sequent of the form $\langle \Gamma, \alpha \rangle$. Given that we are concerned here only with formally valid reasoning, what $E^{u,v}$ tells us is just that, if there are proofs available from α to σ and β to σ, then we have (classically) proofs of $(\alpha \supset \sigma)$, and $(\beta \supset \sigma)$. With these, and the disjunction elimination rule we only have that $(\alpha \vee \beta), (\alpha \supset \sigma), (\beta \supset \sigma) \vdash_L \sigma$. So, the rules are compatible with non-standard valuations, in which $v(\alpha) = v(\beta) = 0$ whilst $v(\alpha \vee \beta) = 1$.

The failure occurs for this form of inferentialism because it has no machinery for dealing with falsity, so we require appeal to the resources of semantics in order to account for the invalidity of sequents. In other words, it looks as though we need some sort of model-checking device to deal with refutation (or falsity). One way of thinking about this is that there is a division of labour between an inferential structure, which is well equipped to provide epistemic traction on the validity of argument, and a semantic structure, which is capable of providing counterexamples. To clarify, we then expect that for every formula α:

Either $\exists \Gamma^+ (\Gamma^+ \vdash_L \alpha)$ or, $\exists V (V \not\models \alpha)$.

Then, for some agent A, concerned with α, A may ask (i) is α valid in an inference structure L?; (ii) is there a countermodel to α in a semantic structure V? The thought here is that we are required to appeal to V in order to account for the possible countermodels to a possibly valid formulas, since, L is incapable of this, by itself.

1.2. **Possible solutions.** It is well known that this unwanted feature does not carry over to multiple-conclusion inference structures, such as Gentzen's LK. The obvious reason for this, which is brought to the fore in Restall's recent work [**?**], is that these structures provide for a symmetrical way of dealing with proofs and refutations, or asssertion and denial. Restall's approach is connected to a wider body of literature in which inference structures are considered to be constitutive of normative constraints over the coherence of agents' commitments [**? ? ?**]. For example:

> [I]nferential rules do not prescribe what ought to be done, but what is allowed and what is not allowed to do when one asserts or judges that p. Inferential rules do not primarily consist in commands,

> obligations or incentives for speakers or believers; they rather constrain our linguistic practices by delimiting what, on an inferential point of view, we may and may not do by entertaining conceptual contents. [**?**]

On this view, a valid inference in L is understood to be a constraint upon the coherent assertion and denial of sentences. For example, an agent asserting $\alpha, \beta \vdash_L \alpha \wedge \beta$, may be said to be rationally committed to not simultaneously asserting α, β and denying $\alpha \wedge \beta$. Importantly, this view takes logical consequence to tell us not just about assertion, but about both assertion and denial, where denial is taken to be conceptually prior to negation. Restall proposes an inferentialist account of the classical sequent calculus in these terms: If $\Gamma \vdash \Delta$, then it is incoherent to assert all of Γ, and deny all of Δ. As is clear, dealing with assertion and denial symmetrically is also reflected in the symmetrical construction of the sequent calculus.[3] A construction which allows sets of formulas to appear on either side of $\vdash$ lends itself to the view of inferential rules as providing pragmatic constraints over legitimate combinations of sentences.

However, there are several issues with such multiple-conclusion structures, which are well documented in the relevant literature. The most prominent of these is that they fail to adequately capture the structure of ordinary argument, which is a key desiderata for the inferentialist. Steinberger [**?**] puts this as follows:

> Principle of answerability: only such deductive systems are permissible from the inferentialist point of view as can be seen to be suitably connected to our ordinary deductive inferential practices.

As suggested above, this principle does not amount to an over-simplistic identification of inference rules with actual ordinary language practice, since these are also taken to be normative constraints on the coherence of assertions and denials. Nonetheless, the basic structure of inference is one that is typically construed in argument form from (possibly) multiple premises, and a single-conclusion. As Rumfitt [**?**] has it;

> [...] the rarity, to the point of extinction, of naturally occurring multiple-conclusion arguments has always been the reason why mainstream logicians have dismissed multiple-conclusion logic as little more than a curiosity. (79)

Restall's account, and his brand of inferentialism, goes some way to alleviating these concerns. Even so, there is a deeper point that these arguments from ordinary practice allude to which has to do with the lack of any independent account

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[3] Though Peregrin, for example, does not follow suit: '[$\Gamma \vdash_L \alpha$] is much more reasonably construed as a constraint: the exclusion of the possibility to deny α when one has asserted Γ' [**?** , p.3].

of assertion or denial. For instance, intuitionism provides an account of the assertibility conditions for a set of formulas, with the general condition that a formula α is assertible whenever there is a proof of α. Treating denial symmetrically suggests the need for an account of the deniability conditions for a set of formulas, perhaps with the general condition that a formula α is deniable whenever there is a refutation of α. This would then be in keeping with the inferentialist motto that the meaning of a logical constant consists in its assertibility conditions, together with its deniability conditions. But, pursuing this kind of project requires a rather different approach to Restall's, since his notion of the coherence of assertions and denials would then be derivative of these explanatory structures. To this end, I pursue this project in an abstract setting, by first defining an inference structure consisting of a dual-calculus of assertions and denials whose inference relations are both in single-conclusion form. I then go on to develop a generalised approach to inferential semantics, which illuminates the expressive superiority of this approach to the multiple-conclusion structure, though the two are shown to be structurally equivalent. To finish, I return to Restall's concern with classical inference structures, showing that, in fact, Restall's account can be better put in terms of a dual-calculus of intuitionistic and co-intuitionistic logic. In essence, if we pair intuitionistic logic with its dual, then we get a form of constructively classical logic which is profitably interpreted in a Nelson-style semantics.

2. Duality and coherence

2.1. **Proof and refutation structures.** Consider a proof in some inference structure L, $\Gamma \vdash_L \alpha$ as showing that the combined assertion of $\Gamma \cup \alpha$ is coherent. This follows from the typical definition of closure under the inference relation $\vdash_L$: $\alpha \in \Gamma \leftrightarrow \Gamma \vdash_L \alpha$. Equally, we can consider a refutation in the dual structure L_D, $\alpha \dashv_{L_D} \Gamma$ as showing that the combined denial $\Gamma \cup \alpha$ is coherent. Again, we can define closure under this dual relation $\dashv_{L_D}$: $\alpha \in \Gamma \leftrightarrow \alpha \dashv_{L_D} \Gamma$. We can also think of this pair of inference relations as determining a complete inference structure, where now $L = \langle S, \vdash_L, \dashv_L \rangle$. Then, $\vdash_L$ preserves the set of coherent assertions, Γ^+, and $\dashv_L$ preserves the set of coherent denials, Δ^-. So, we can define a refutation inference, $\alpha \dashv_L \Delta$, to be interpreted as showing that whenever all of Δ is deniable, α is deniable also (in L).

The account of refutation given in the following closely follows those given in [? ? ? ? ?], whilst the abstract account of inference structures follows that given in [?].

Definition 2. (Inference structure) Define an inference structure L as an ordered pair, $\langle S, \vdash_L \rangle$, where S is as above, and $\vdash_L$ is a binary derivability relation between a subset of formulas of S (denoted (Γ, Δ)), and formulas of S, (denoted (α, β)), so $\vdash_L \subseteq \mathcal{P}(S) \times S$.[4] A sequent in L is an ordered pair $\langle \Gamma, \alpha \rangle$ where Γ is a set of formulas and α a single formula (and $\Gamma \cup \{\alpha\}$).

We say $\vdash_L$ is normal when the following conditions hold, for all formulas, $\alpha, \beta \in S$, and all subsets $\Sigma, \Delta \subseteq S$:
$(\mathbb{R}) : \alpha \vdash_L \alpha$
$(\mathbb{M}) : \Sigma \vdash_L \alpha$ implies $\Sigma, \Delta \vdash_L \alpha$
$(\mathbb{T}) :$ If $\Sigma \vdash_L \alpha$ for each $\alpha \in \Delta$, and $\Sigma, \Delta \vdash_L \beta$, then $\Sigma \vdash_L \beta$.

Definition 3. An arbitrary relation on S, $\vdash_L \subseteq \mathbb{P}(S) \times S$ is finitary, where, for all $\Gamma, \alpha \in S$, if $\Gamma \vdash_L \alpha$, then there is a finite $\Gamma' \subseteq \Gamma$ where $\Gamma' \vdash_L \alpha$.

Definition 4. (Inference rule) An n-premise rule R in L is an $n+1$-ary relation on the sequents of L. Where a rule is associated with a connective $\#$, the set of formulas will be closed under the operation $\#(\alpha_1, ..., \alpha_n)$, such that, when all $(\alpha_1, ..., \alpha_{n+1}) \in \#$, (and $\{\alpha_1, ..., \alpha_n\} \subseteq S$), $\alpha_{n+1} \in S$.

A set of inference rules, by defining a basic set of inferences also establish a wider set of inferences over the formulas of S. The rules of a proof-theory thus constrain L by defining the set of formulas closed under the derivability relation. The closure of a set of formulas under rules R and the structural constraints above tells us that when α is provable from Γ by means of R, then it is the case that either there is a sequence $\alpha_1, \alpha_2, ..., \alpha_n$, where $\alpha_n = \alpha$ and each $\alpha_i \in \{\alpha_1, ...\alpha_{n-1}\}$ is either an element of Γ, or there is some $\Delta \vdash_L \alpha_i \in R$ where $\Delta \subseteq \{\alpha_1, ...\alpha_{i-1}\}$. This requires that $\vdash_L$ is finitary, but that demand follows from the the requirement that a good proof-system should be finitary given that it is an essential part of ordinary reasoning that it is finitary.

Call a logic L that is constrained in this way, proof-theoretically defined, when L distributes S on $\mathscr{L}$ in terms of the collection of sequents that are provable according to L.

Definition 5. Any sequent $\langle \Gamma, \alpha \rangle \in L$ is L-valid, then $\Gamma \vdash_L \alpha$.

In foregrounding the notion of proof, we are interested in the conditions under which α is is provable given a set of assumptions Γ, thus allowing for a highly non-trivial account of the assertibility of α in the context of the set of assumptions Γ. The dual notion of refutation is, for our purposes at least, simple to define.

[4]Obviously, we are working with a formal language here, so we probably don't strictly need to use the phrase "well-formed formula" (wff) of $\mathscr{L}$. I should flag up that I interpret the phrase quite liberally, allowing, for example "$\{\phi|...\}$" to be a wff.

Definition 6. (Duality) The dual of $\Gamma \vdash_L \alpha$ is $\alpha \dashv_{L_D} \Gamma$ (for any $\Gamma \subseteq S, \alpha \in S$).

Definition 7. (Dual inference structure) Define a dual inference structure L_D as an ordered pair, $\langle S, \dashv_{L_D} \rangle$, where S is as above, and $\dashv_L$ is a binary derivability relation between a formula and a subset of formulas of S, so $\dashv_{L_D} \subseteq S \times \mathcal{P}(S)$.[5] Say that $\dashv_{L_D}$ is normal when it is reflexive, transitive, monotonic and finitary (the definitions given for $\vdash_L$ carry over striaghtforwardly). A sequent in L is an ordered pair $\langle \alpha, \Gamma \rangle$ where Γ is a set of formulas and α a single formula (and $\Gamma \cup \{\alpha\}$). The definition of inference rules also carries over, bearing in mind the reversal of $\dashv_{L_D}$.

Then, $\langle S, \dashv_{L_D} \rangle$ is a refutation structure for $\langle S, \vdash_L \rangle$ when, for any non-theorem of $\langle S, \vdash_L \rangle$, $\emptyset \not\vdash_L \alpha$, then $\alpha \dashv_{L_D} \emptyset$. In foregrounding the notion of refutation, we are interested in the conditions under which α is is refutable given a set of (denied) assumptions Γ, thus allowing for a highly non-trivial account of the deniability of α in the context of Γ.

The relation between the two structures is simpler with the following definition:

Definition 8. (Closure) Say that a theory $\Sigma \subseteq S$ is $\vdash_L$-closed, for some L, when, for all $\Sigma \vdash_L \alpha$, $\alpha \in \Sigma$. Say that a theory $\Theta \subseteq S$ is $\dashv_{L_D}$ closed for some L_D, when, for all $\alpha \dashv_{L_D} \Theta$, $\alpha \in \Theta$.

Letting Γ^+ denote the set of L-valid formulas, and Δ^- the set of L_D-valid formulas, we say that $\langle S, \dashv_{L_D} \rangle$ is a refutation structure for $\langle S, \vdash_L \rangle$ whenever $\Delta^- = S \setminus \Gamma^+$. That is to say, whenever Δ^- is equivalent to the set of non-derivable formulas for L. Then, we can think of a sequent with an empty *l.h.s*, $\emptyset \vdash_L \alpha$, as indicating that an agent may coherently assert α under any circumstance; dually, we can think of a sequent with an empty *r.h.s*, $\alpha \dashv_L \emptyset$, as indicating that an agent may coherently deny α under any circumstance. With this in mind, the coherence constraint over the relationship between $\vdash_L$ and $\dashv_L$ is the following.

Corollary 9. *(Coherence) For no atomic formula $\alpha \in S$ is it the case that $\emptyset \vdash_L \alpha$ and $\alpha \dashv_L \emptyset$ whenever $\vdash_L$, $\dashv_L$ are normal.*

This is equivalent to Restall's suggestion that we interpret validity in terms of the preclusivity of an agent A simultaneously asserting α and denying α under any circumstances. With this in place, we can define the combined inference structure:

Definition 10. (Combined inference structure) Define a combined inference structure L_C as an ordered triple, $\langle S, \vdash_{L_C}, \dashv_{L_C} \rangle$ satisfying the conditions given above.

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[5] Obviously, we are working with a formal language here, so we probably don't strictly need to use the phrase "well-formed formula" (wff) of $\mathscr{L}$. I should flag up that I interpret the phrase quite liberally, allowing, for example "$\{\phi|...\}$" to be a wff.

We can also give the following natural constraint over the combined inference structure:

Definition 11. (Totality) Say that a combined inference structure L_C is total whenever, for any formula α, either $\emptyset \vdash_{L_C} \alpha$ or $\alpha \dashv_{L_C} \emptyset$, and for no α is it the case that both $\emptyset \vdash_{L_C} \alpha$ and $\alpha \dashv_{L_C} \emptyset$.

2.2. **Multiple-conclusion inference structures.** Before developing an abstract approach to inferentialist semantics in the context of combined inference structures, we pause to note their relation with multiple-conclusion structures.

Definition 12. (Multiple-conclusion inference structure) We define a multiple-conclusion inference structure L_M as an ordered pair, $\langle S, \mid\!\frac{\quad}{L_M}\!\mid \rangle$, where S is the enumerable set of formulas in a language $\mathscr{L}$, and $\mid\!\frac{\quad}{L_M}\!\mid$ is a binary derivability relation between a subset of formulas of S (denoted (Γ, Δ)), and a subset of formulas of S, such that $\mid\!\frac{\quad}{L_M}\!\mid \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$, (elements of S are single wff's denoted (α, β)).[6] A sequent is an ordered pair $\langle \Gamma, \Delta \rangle$ where Γ, Δ are sets of formulas of S. As usual, we write a sequent as $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \Delta$.

Say that $\mid\!\frac{\quad}{L_M}\!\mid$ is normal when the following conditions hold for all formulas, $\alpha, \beta \in S$, and all subsets $\Sigma, \Delta, \Gamma, \Theta \in S$:
$(\mathbb{R}) : \Gamma \cap \Delta \neq set$ then $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \Delta$
$(\mathbb{M}) : \Sigma \mid\!\frac{\quad}{L_M}\!\mid \Delta$ implies $\Sigma, \Gamma \mid\!\frac{\quad}{L_M}\!\mid \Delta, \Theta$
$(\mathbb{T})$: If $\Gamma, \alpha \mid\!\frac{\quad}{L_M}\!\mid \Delta$ and $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \alpha\Delta$ then $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \Delta$.

Furthermore, say that an inference structure is finitary, when, for all $\Gamma, \Delta \subseteq S$, if $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \Delta$, then there are finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ where $\Gamma' \mid\!\frac{\quad}{L_M}\!\mid \Delta'$.

Proposition 13. *The relational product of the closure of S under $\vdash_{L_C}$ and $\dashv_{L_C}$ is a multiple conclusion inferential structure $\langle S, \mid\!\frac{\quad}{L_M}\!\mid \rangle$.*

To see why this is the case, recall the definition of closure under inference rules given above. We first show that, whenever $\Gamma \vdash_{L_C} \alpha$, $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \alpha$, which requires only that the multiple-conclusion versions of R, M, T preserve the first derivation. If Γ is empty then this is obvious, as is the case where $\alpha \in \Gamma$. Then, we say that the multiple-conclusion relation $\mid\!\frac{\quad}{L_M}\!\mid$ contains the corresponding relation $\vdash_{L_C}$, since $\Gamma \vdash_{L_C} \beta$ iff $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \{\beta\}$ for $\Gamma \in \mathbb{P}(S)$, and $\beta \in S$. Similarly, $\alpha \dashv_{L_C} \Delta$ iff $\{\alpha\} \mid\!\frac{\quad}{L_M}\!\mid \Delta$ for $\Delta \in \mathbb{P}(S)$, and $\alpha \in S$. These are the least single-conclusion relations contained in the multiple-conclusion relation $\mid\!\frac{\quad}{L_M}\!\mid$ [**?**]. Consider the case in which $\Gamma \vdash_{L_C} \alpha$, and, by definition, $\alpha \dashv_{L_C} \Gamma$ is a refutation in L_C, to check T. Say that $\Gamma_1, \beta \mid\!\frac{\quad}{L_M}\!\mid \alpha$ and $\Gamma_2 \mid\!\frac{\quad}{L_M}\!\mid \beta$ where $\Gamma = \Gamma_1 \cup \Gamma_2$. By induction, we have $\alpha \mid\!\frac{\quad}{L_M}\!\mid \Gamma_1 \cup \beta$ and

[6] I use the non-standard notation $\mid\!\frac{\quad}{L_M}\!\mid$ rather than $\vdash_{L_M}$ both to note their difference, and also because later I shall show the equivalence between these structures and the combined inference structure.

$\beta \mid\!\frac{\quad}{L_M}\!\mid \Gamma_2$ as refutations in L_M. Assume that $\emptyset \mid\!\frac{\quad}{L_M}\!\mid \alpha$. Then, either (a) $\emptyset \mid\!\frac{\quad}{L_M}\!\mid \beta$, or (b) $\emptyset \mid\!\frac{\quad}{L_M}\!\mid \theta$, for some $\theta \in \Gamma_1$. For (a), it is obvious that $\emptyset \mid\!\frac{\quad}{L_M}\!\mid \theta$, for some $\theta \in \Gamma_2$, so, in either case we have that $\emptyset \mid\!\frac{\quad}{L_M}\!\mid \theta$ for some $\theta \in \Gamma$, which obeys the definitions given in [**?**] of least single-conclusion relations.

However, this does provide us with a slightly different interpretation of multiple-conclusion inference structures, that, nonetheless is entirely compatible with Restall's.

Definition 14. (General coherence) A pair of sets of formulas $\langle\Gamma, \Delta\rangle$ is coherent iff $\Gamma \cap \Delta \neq \emptyset$, then $\Gamma \mid\!\frac{\quad}{L}\!\mid \Delta$. A pair of sets of formulas $\langle\Gamma, \Delta\rangle$ is incoherent iff $\Gamma \cap \Delta = \emptyset$, then $\Gamma \not\mid\!\frac{\quad}{L}\!\mid \Delta$.

Then, given the way in which we have constructed a multiple-conclusion structure as the relational product of a combined inference structure, we can read any sequent $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \Delta$ (rewritten $\alpha_1, ..., \alpha_n \mid\!\frac{\quad}{L_M}\!\mid \beta_1, ..., \beta_m$) as: either $\alpha_1 \wedge ... \wedge \alpha_n \mid\!\frac{\quad}{L} \beta_i$ (where β_i is some formula in Δ), or $\alpha_i \frac{\quad}{L}\!\mid \beta_1 \wedge ... \wedge \beta_m$ (where α_i is some formula in Γ).

We can render the above relation more perspicuous if we consider the construction of bipartitions over the set of formulas S, which is also the first step in the construction of a semantic structure as determined by an inference structure. The following purposefully combines proofs in terms of multiple-conclusion inference structures with combined inference structures in order to clarify their relationship.

Definition 15. (Closure) Say that a theory $\Sigma \subseteq S$ is $\mid\!\frac{\quad}{L_M}\!\mid$-closed, for some L, when, for all $\langle\Gamma, \Delta\rangle \in L$, $\Delta \cap \Sigma \neq \emptyset$ whenever $\Gamma \subseteq \Sigma$.

Theorem 16. *(Coherence extension) $\langle\Gamma, \Delta\rangle$ is coherent, so $\Gamma \mid\!\frac{\quad}{L_M}\!\mid \Delta$, iff (i). For any extension, Γ_i, of Γ, there exists an atomic element $\alpha \in \Delta$, such that $\Gamma_i = \Gamma \cup \{\alpha\}$; (ii). For any extension, Δ_i, of Δ, there exists an atomic element $\beta \in \Gamma$, such that $\Delta_i = \Delta \cup \{\beta\}$; (iii). For every $\langle\Gamma, \Delta\rangle$, there exists maximal extensions $\langle\Gamma', \Delta'\rangle$ such that $\{\Gamma'\} \cup \{\Delta'\} = \emptyset$, so $\Gamma' \not\mid\!\frac{\quad}{L_M}\!\mid \Delta'$.*

We can think of these maximal extensions as bipartions (for L_M), $\langle\Gamma^i, \Delta^i\rangle$, where $\Gamma^i \not\mid\!\frac{\quad}{L_M}\!\mid \Delta^i$, and Δ^i is the complement of Γ^i. We will go on to show that the subset $\Gamma^i \subseteq S$ is closed iff Γ^i is closed under $\mid\!\frac{\quad}{L_C}$, and the complement Δ^i is closed under $\frac{\quad}{L_C}\!\mid$ for any L_C, which determines L_M.

Lemma 17. *For a finite normal multiple-conclusion inference structure L_M (equivalently a combined inference structure L_C), and for any pair of theories $\langle\Gamma, \Delta\rangle$ (in S), where $\Gamma \not\mid\!\frac{\quad}{L_M}\!\mid \Delta$, there exists a bipartition extending $\langle\Gamma, \Delta\rangle$ to Γ', and its complement, Δ', where $\Delta' =_{df} \forall\alpha \notin \Gamma'$, with $\Gamma' \cup \Delta' = S$, and $\Gamma' \cap \Delta' = \emptyset$.*

Proof. Take the enumerable formulas of S, $\{\alpha_1, \alpha_2 ... \alpha_i, \alpha_{i+1}\}$, and any Γ, Δ for which $\Gamma \not\mid\!\frac{\quad}{L_M}\!\mid \Delta$, and construct the following chain of pairs of theories:

i. $\langle \Gamma_n, \Delta_n \rangle = \langle \Gamma, \Delta \rangle$

ii. $\langle \Gamma_{n+1}, \Delta_{n+1} \rangle = \begin{cases} \langle \Gamma_n \cup \{\alpha_{i+1}\}, \Delta_n \rangle \text{ if } (\Gamma_n \vdash_{L_M}\!\dashv \Delta_n, \alpha_{i+1}) \\ \langle \Gamma_n, \Delta_n \cup \{\alpha_{i+1}\} \rangle \text{ if } (\Gamma_n, \alpha_{i+1} \vdash_{L_M}\!\dashv \Delta_n) \end{cases}$

The limit of this construction is:

$\langle \Gamma' \rangle = \bigcup_{n \in I} \Gamma_n$, and $\langle \Delta' \rangle = \bigcup_{n \in I} \Delta_n$.

The pair $\langle \Gamma', \Delta' \rangle$ form a bipartition over formulas of S, with $\Gamma' \not\vdash_{L_M}\!\dashv \Delta'$. Then, (by the fact that L is finitary), for any subsets of $\langle \Gamma', \Delta' \rangle$, we have $\Gamma \not\vdash_{L_M}\!\dashv \Delta$, when $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$. □

So, we can construct an arbitrary bipartition (for L) recursively as:

$\langle \Gamma^i, \Delta^i \rangle$ iff, for all $\alpha \in S$
$\alpha \in \Gamma^i$ when $\langle \Gamma^i \vdash_{L_M}\!\dashv \alpha, \Delta^i \rangle$
$\alpha \in \Delta^i$ when $\langle \Gamma^i, \alpha \vdash_{L_M}\!\dashv \Delta^i \rangle$.

This requires that, at every stage of the construction, Γ^i keeps track of the set of formulas that are derivable from Γ, and Δ^i the set of propositions being such that Δ is derivable from them. So, we think of an arbitrary bipartition $\langle \Gamma^i, \Delta^i \rangle$, Γ^i as denoting a complex set of coherent assertions in the context of that bipartition (given L), and Δ^i a set of coherent denials. To render this perspicuous, we will denote the bipartition $\langle \Gamma^+, \Delta^- \rangle$ in what follows. The addition of an atomic to a stage of the construction of a bipartition indicates that the agent takes a settled stance towards it. Then, from the point of view of the combined inference structure L_C, which determines L_M, for a pair of subsets of formulas $\langle \Gamma, \Delta \rangle$ and an atomic formula α, it must be the case that *either* $\Gamma \vdash_{L_C} \Delta, \alpha$ *or* $\Gamma, \alpha \;{}_{L_C}\!\dashv \Delta$. The first tells us what it is possible to assert on the assumption that we already assume Γ to be assertible, and the latter, what it is possible to deny on the basis of assuming Δ to be deniable.

With this in mind, we have the following result.

Lemma 18. *If* $\emptyset \vdash_{L_C} \alpha$ *for all* $\alpha \in \Gamma$, *and* $\beta \;{}_{L_C}\!\dashv \emptyset$ *for all* $\beta \in \Delta$, *then* $\Gamma \not\vdash_{L_M}\!\dashv \Delta$. *And, if* $\Gamma \not\vdash_{L_M}\!\dashv \Delta$, *then* $\emptyset \vdash_{L_C} \alpha$ *for all* $\alpha \in \Gamma$, *and* $\beta \;{}_{L_C}\!\dashv \emptyset$ *for all* $\beta \in \Delta$.

Proof. The proof is fairly obvious by the above definition of bipartitions, so I show just one instance. Assume that $\emptyset \vdash_{L_C} \alpha$ for all $\alpha \in \Gamma$ and $\beta \;{}_{L_C}\!\dashv \emptyset$ for all $\beta \in \Delta$, but $\Gamma \vdash_{L_M}\!\dashv \Delta$. It follows that $\emptyset \vdash_{L_C} \alpha_i$, in which case, $\Gamma \setminus \alpha_i \vdash_{L_C} \alpha_i, \Delta$ (by $\mathbb{M}$), and, clearly, $\Gamma \setminus \alpha_i, \alpha_i \vdash_{L_M}\!\dashv \Delta$. Then $\Gamma \setminus \alpha_i \vdash_{L_M}\!\dashv \Delta$ (by $\mathcal{T}$). But, by the definition of bipartitions, we have $\{\Gamma \setminus \alpha_i \cup \Delta\} \subseteq \{\Gamma \cup \Delta\}$, so $\Gamma \setminus \alpha_i \not\vdash_{L_M}\!\dashv \Delta$. Conversely, suppose that $\Gamma \not\vdash_{L_M}\!\dashv \Delta$, but either $\emptyset \not\vdash_{L_C} \alpha$ for some $\alpha \in \Gamma$, or $\beta \;{}_{L_C}\!\not\dashv \emptyset$ for some $\beta \in \Delta$. Then, either $\alpha \;{}_{L_C}\!\dashv \emptyset$ for some $\alpha \in \Gamma$, or $\emptyset \vdash_{L_C} \beta$ for some $\beta \in \Delta$, (in which case $\Gamma \vdash_{L_M}\!\dashv \Delta$ (by $\mathbb{M}$). □

3. From Inference Structures to Semantic Structures

We now turn to discuss inferentialist semantics. First, recall from the introduction the view that for every atomic $\alpha \in S$:

Either $\exists\Gamma^+(\Gamma^+ \vdash_L \alpha)$ or, $\exists V(V \not\models \alpha)$.

As we saw, a standard inference structure is not strong enough to dispell the requirement of appeal to the right hand side. However, in the context of a combined inference structure (equivalently a multiple-conclusion inference structure), this is no longer the case since we have a combined structure of proofs and refutations which provide the means to account for refuted formulas without appeal to model theory. Crucially, by the definition of bipartitions, we know that, at any stage of their construction where $\langle\Gamma_i, \Delta_i\rangle \subseteq \langle\Gamma^+, \Delta^+\rangle$, we have $\Gamma_i \cap \Delta^i = \emptyset$, and so $\Gamma_i \not\vdash\!\dashv_{L_M} \Delta_i$; for any $\alpha \in \Gamma_i$, $\alpha \not\dashv_{L_C} \emptyset$, and vice-versa.

Then (to get ahead of ourselves), we can think of each stage of the construction as constituting a partial interpretation where the formulas of Γ_i are mapped to the valuation $\{1\}$ in a semantic structure, and the formulas in Δ_i to $\{0\}$. To clarify further, we can now rewrite the above so that, for every atomic $\alpha \in S$;

Either $\exists\Gamma^+(\Gamma^+ \vdash_{L_C} \alpha)$ or, $\exists\Delta^-(\alpha \dashv_{L_C} \Delta^-)$.

Thus, the proposition that we want is this:

Proposition 19. *Say that $\Gamma \vdash\!\dashv_{L_M} \Delta$ is correct on a bipartition $\langle\Gamma^+, \Delta^-\rangle$ iff either some $\alpha \in \Gamma$ is in Δ^-, or some $\beta \in \Delta$ is in Γ^+. Equivalently, either $\alpha \dashv_{L_C} \Delta^-$, or $\Gamma^+ \vdash_{L_C} \beta$.*

Lemma 20. *For a multiple-conclusion inference structure L_M defined as above, there exist (maximal) bipartitions $\langle\Gamma^+, \Delta^-\rangle$ such that, for every $\alpha \notin \Gamma^+$, $\alpha \in \Delta^-$, and, where every $\alpha \notin \Gamma^+$ (equivalently, $\alpha \in \Delta^-$), $\alpha \dashv_{L_C} \emptyset$, and, every $\alpha \in \Gamma^+$, $\emptyset \vdash_{L_C} \alpha$.*

Proof. Again, the proof is fairly obvious by the construction of bipartitions. We know already that, if Γ^+ is closed under L_M, then $\Gamma^+ = S \setminus \Delta^-$. Then, for any $\alpha \notin \Gamma^+$, we have $\alpha \vdash\!\dashv_{L_M} \emptyset$, (equivalently $\alpha \dashv_{L_C} \emptyset$). For this, just note that, if $\Gamma^i \subseteq \Gamma^+$, then $\Delta^{i+1} \cap \Gamma^+ \neq \emptyset$, where $\Delta^{i+1} = \Delta^i \cup \{\alpha\}$ so either $\alpha \in \Gamma^+$, or $\Delta^i \cap \Gamma^+ \neq \emptyset$. If the former ($\alpha \in \Gamma^+$), then $\Gamma^i \cup \alpha \subseteq \Gamma^+$, so $\Delta^i \cap \Gamma^+ \neq \emptyset$. Suppose, instead that Γ^+ is closed under L_M, but Γ^+ is not maximal. Then, there must be some theory $\Gamma^i \supset \Gamma^+$ (also closed under L_M), where there is a formula $\alpha \in \Gamma^i \setminus \Gamma^+$. But, since Γ^i is closed under L_M, $\Gamma^i \subseteq \Gamma^+$ and so $\alpha \notin \Gamma^i$. □

Lemma 21. *Any finite normal multiple-conclusion inference structure (equivalently, combined inference structure) is sound and complete with respect to its maximal bipartitions.*

Proof. First, define the subset of maximal bipartitions of S that is determined by an inference structure L as $B(L) = \left\{\langle\Gamma^+, \Delta^-\rangle : \Gamma^+ \underset{L_M}{\not\vdash\!\dashv} \Delta^-\right\}$. Then, suppose that $\Gamma \underset{L_M}{\not\vdash\!\dashv} \Delta$. We know, by the above, that, for any $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^-$, $\Gamma^i \underset{L_M}{\not\vdash\!\dashv} \Delta^i$. This gives us completeness. For soundness, consider some $\Gamma \underset{L_M}{\vdash\!\dashv} \Delta$, to show that there is no $B \in B(L)$ for which $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^-$. Suppose otherwise, then, since $\Gamma^+ \underset{L_M}{\not\vdash\!\dashv} \Delta^-$, by $\mathbb{M}$, we have $\Gamma \underset{L_M}{\not\vdash\!\dashv} \Delta$, contrary to our assumption.

It is simple to see that this gives us proposition 19. Take an arbitrary sequent, $\langle\Gamma, \Delta\rangle$, which we rewrite as $\{\alpha_1, ..., \alpha_n\} \underset{L_M}{\vdash\!\dashv} \{\beta_1, ..., \beta_m\}$. Then, $\langle\Gamma, \Delta\rangle$ is valid for a logic L_C when ($\alpha_1 \underset{L_C}{\dashv} \Delta^-$ or $\alpha_2 \underset{L_C}{\dashv} \Delta^-$ or...$\alpha_m \underset{L_C}{\dashv} \Delta^-$) or ($\Gamma^+ \underset{L_C}{\vdash} \beta_1$ or $\Gamma^+ \underset{L_C}{\vdash} \beta_2$ or...$\Gamma^+ \underset{L_C}{\vdash} \beta_m$). Then, $\Gamma \underset{L_M}{\vdash\!\dashv} \Delta$ is correct on the partition $\langle\Gamma^+, \Delta^-\rangle$ iff either some $\alpha \in \Gamma$ is in Δ^-, or some $\beta \in \Delta$ is in Γ^+. □

3.1. **Semantics and Galois connections.** It is simple to define the set of truth-value assignments that are consistent with an inference structure L to be the set of characteristic functions associated with its maximal bipartitions. In the following, since we dealing with bipartitions we will assume that $\mathcal{V} = \{1, 0\}$. Then, we define the characteristic functions for bipartions as follows.

Definition 22. (Characteristic function) For any relatively maximal theory, Γ', we take a valuation to be the characteristic function of Γ' such that:

$v(\Gamma') = \{v \in U : v(\alpha) = 1 \text{ for each } \alpha \in \Gamma'\}$;

and, for the complement

$v(\Delta') = \{v \in U : v(\alpha_i) = 0 \text{ for each } \alpha_i \in \Delta'\}$.

Lemma 23. *For a set of valuations $V \subseteq U$, with a valuation v defined over formulas $\alpha \in S$ as $\{v \in U : \alpha \to \{1, 0\}\}$, $v(\alpha) = 1 \leftrightarrow \emptyset \underset{L_C}{\vdash} \alpha$, $v(\alpha) = 0 \leftrightarrow \alpha \underset{L_C}{\dashv} \emptyset$; and $\Gamma \underset{L_M}{\vdash\!\dashv} \Delta$ iff either, there is some $\alpha \in \Gamma$ and $v(\alpha) = 0$, or there is some $\beta \in \Delta$ and $v(\beta) = 1$.*

Proof. On the above construction, we have defined maximal bipartitions over S, $\langle\Gamma^+, \Delta^-\rangle$, where $\Gamma^+ \cup \Delta^- = S$, and $\Gamma^+ \cap \Delta^- = \emptyset$. We show that the set of such valuations V is consistent with L. Suppose otherwise, then there must exist two sets of formulas, Σ, Φ, with $\Sigma \underset{L_M}{\vdash\!\dashv} \Phi$, and, for some $v \in V$, $v(\Sigma) = 1$ whilst $v(\Phi) = 0$. In other words, on the above construction, it must be that $\Sigma \subseteq \Gamma^+$, whilst $\Phi \subseteq \Delta^-$, whilst $\Sigma \underset{L}{\vdash\!\dashv} \Phi$, but this contradicts the construction of maximal bipartitions. □

We can go further than this by noting that the relation between valuation spaces and logics induces a Galois-connection [? ? ? ?]. Formally, we define the map sending $L \to V$, denoted $\mathbb{V}(L) =_{df} \Gamma \underset{L}{\vdash\!\dashv} \Delta : \forall v \in \mathbb{V}(L)\,\{v(\Gamma) = 0 \text{ or } v(\Delta) = 1\}$. In the other direction, the map $V \to L$, denoted $\mathbb{L}(V) =_{df} v \in V : \forall\langle\Gamma, \Delta\rangle \in L\,\{v(\Gamma) = 0 \text{ or } v(\Delta) = 1\}$. Then, the pair $\{\mathbb{L}, \mathbb{V}\}$ form an antitone Galois connection between the power-set of U, and the power-set of L, both ordered by inclusion.

The following assumes that any inference structure is in the form of either L_M or L_C. This in hand, it is simple to define closure operations over:

(P1): The set of all valuation-spaces $V \subseteq U$ on $\mathscr{L}$, ordered by set-inclusion;

(P2): The set of all inference structures $L \subseteq L'$ on $\mathscr{L}$, ordered by set-inclusion.

Definition 24. Define a closure operator cl as a function on the posets $\langle V, L\rangle$, where cl obeys the following clauses for all x, y on $\langle V, L\rangle$:

(c1) $x \leq cl(x)$

(c2) $cl(cl(x)) \leq cl(x)$

(c3) $x \leq y \rightarrow cl(x) \leq cl(y)$

This ensures that, where cl is a closure operator on a poset $\langle P, \leq\rangle$, and x is an element of P, then x is closed iff $cl(x) = x$. In our context, this gives us an abstract completeness theorem over $\langle \mathbb{V}, \mathbb{L}\rangle$.

Fact 25. *For each $V \subseteq U$ and $L \subseteq L'$ (for some S):*

$$L \subseteq \mathbb{L}(\mathbb{V}(L)) \tag{3.1}$$

$$V \subseteq \mathbb{V}(\mathbb{L}(V)) \tag{3.2}$$

$$L \subseteq L' \Rightarrow \mathbb{V}(L') \subseteq \mathbb{V}(L) \tag{3.3}$$

$$V \subseteq U \Rightarrow \mathbb{L}(U) \subseteq \mathbb{L}(V) \tag{3.4}$$

Proof. Given at length in [**?**]. □

(1.5) indicates that when we determine $\mathbb{V}(L)$, and then induce an inference structure $\mathbb{L}$ from the valuation space determined, then $\mathbb{L}$ will contain L. Similarly, (1.6) tells us that, when we determine $\mathbb{L}(V)$, and then determine a valuation space $\mathbb{V}$ from the inference structure determined, that $\mathbb{V}$ will contain V. With this, we can formulate abstract soundness and completeness theorems, which, following Dunn and Hardegree [**?**], we call *absoluteness*.

Fact 26. *[**?**] For any L, V;*

- L is absolute iff $L = \mathbb{L}(\mathbb{V}(L))$
- V is absolute iff $V = \mathbb{V}(\mathbb{L}(V))$.

Proof. By the fact that L, V form a Galois map, and the definition of Galois closure (c1-3). □

Resultantly, we can give general soundness and completeness theorems.

Corollary 27. *In general, for any semantic structure $V \subseteq U$, built-up as above, $V = \mathbb{V}(\mathbb{L}(V))$.*

Proof. [**?** , p.200] We prove contrapositively by defining a valuation $v_0 \notin V$ (in order to show that $v_0 \notin \mathbb{V}(\mathbb{L}(V))$). Then define $T = \{\alpha \in S : v_0(\alpha) = 1\}$ and $F = \{\alpha \in S : v_0(\alpha) = 0\}$. For any $v \in V$, $v \neq v_0$, so either $v(\alpha) = 0$ for some $\alpha \in T$ or $v(\alpha) = 1$ for some $\alpha \in F$. Then v satisfies $T \vdash\!\dashv_L F$, and it follows that $T \vdash\!\dashv_L F$ is correct on V. But, by definition, v_0 refutes $T \vdash\!\dashv_L F$, so $v_0 \notin \mathbb{V}(\mathbb{L}(V))$. □

The above tells us that our combined (or multiple-conclusion) inference structures and semantic closure are two sides of the same coin. In effect, this is precisely what we knew at the start of this investigation with respect to classical multiple-conclusion inference structures being capable of dealing with non-standard valuations. Now, however, we have both an abstract account of this, and also a better grasp on its machinery through the purely inferential characterisation of a combined proof and refutation structure.

4. Duality for Restall's Classical Structure

Let us now return to Restall's characterisation of classical inferential structures in this context. Recall that our guiding desiderata with respect to the above interpretation of multiple-conclusion inference structures in terms of combined structures of proofs and refutations was that the notion of coherence in Restall's account looks to be derivative of that of proof and refutation. If we consider the multiple-conclusion inference structure characterised by LK in terms of its internal duality, we may prise it apart into a combined structure. However, rather than letting the relation $\vdash_{LC}$ be characterised by a set of classical rules, we can, in fact, construct $\vdash_{LC}$ by means of the intuitionistic structure LJ, which simply limits the structure of sequents to single-succedents. For $\dashv_{LC}$, we can do precisely the opposite by limiting the structure of sequents to single-antecedents in the co-intuitionistic structure LDJ, though, as above, since we are thinking of $\dashv_{LC}$ as a refutation structure this also amounts to limiting to single-succedents.

We briefly sketch the details, closely following the presentation found in [**?**]. I assume familiarity with the sequent calculus LJ, and present the $\{\wedge, \vee, \neg\}$ fragment of LDJ below:

$$\frac{\alpha \dashv \Gamma}{\alpha \wedge \beta \dashv \Gamma}\ (\wedge L_1) \qquad\qquad \frac{\beta \dashv \Gamma}{\alpha \wedge \beta \dashv \Gamma}\ (\wedge L_2)$$

$$\frac{\theta \dashv \Gamma, \alpha \qquad \theta \dashv \Gamma, \beta}{\theta \dashv \Gamma, \alpha \wedge \beta}\ (\wedge R)$$

$$\frac{\alpha \dashv \Gamma \qquad \beta \dashv \Gamma}{\alpha \lor \beta \dashv \Gamma}\,(\lor L) \qquad \frac{\theta \dashv \Gamma, \alpha}{\theta \dashv \Gamma, \alpha \lor \beta}\,(\lor R_1)$$

$$\frac{\theta \dashv \Gamma, \beta}{\theta \dashv \Gamma, \alpha \lor \beta}\,(\lor R_2)$$

$$\frac{\dashv \Gamma, \alpha}{\neg\alpha \dashv \Gamma}\,(\neg L) \qquad \frac{\alpha \dashv \Gamma}{\dashv \Gamma, \neg\alpha}\,(\neg R)$$

We interpret these, as above, in terms of providing conditions on the refutation of formulas. So, for example, a refutation of $\alpha \land \beta$ consists of a refutation of α or a refutation of β; a refutation of $\alpha \lor \beta$ consists of a refutation of α and a refutation of β; a refutation of $\neg\alpha$ consists of a proof of α.

Remark 28. In fact, negation can be interpreted in various ways, particularly if we develop LDJ to include co-implication as Urbas [**?**] does.[7] However, since we are interested in the interaction between the two structures here, and we already have the constraint that, for any α, either α is provable in LJ or refutable in LDJ, it seems sensible to say that a refutation of $\neg\alpha$ in LDJ consists of a proof of α in LJ. So, we will use negation to "tie" the two structures together to retain the coherence of the combined structure.

Definition 29. Define the duality between the proof and refutation structures defined by LJ and by LDJ with the mapping ^{d}:

$$\begin{gathered}\alpha^d : \alpha \\ (\emptyset \vdash \alpha)^d : \alpha \dashv \emptyset \\ (\alpha \land \beta)^d : \alpha^d \lor \beta^d \\ (\alpha \lor \beta)^d : \alpha^d \land \beta^d \\ (\neg\alpha)^d : \neg(\alpha)^d\end{gathered}$$

We record the following without proof.

Theorem 30. *The mapping* ^{d} *is an involution such that* $\alpha^{dd} = \alpha$. *Moreover,* ^{d} *is an isomorphism such that if* $\Gamma \vdash_{LJ} \alpha$*, then* $\alpha^d \dashv_{LDJ} \Gamma^d$ *[**?**].*

Theorem 31. *The structure* $\langle S, \vdash_{LJ}\rangle$ *contains the same counter-theorems as* $\langle S, \vdash_{LK}\!\dashv \rangle$*. The structure* $\langle S, \dashv_{LDJ}\rangle$ *contains the same theorems as the structure* $\langle S, \vdash_{LK}\!\dashv\rangle$*.*

Proof. The proofs for both are well-known, typically under the titles Glivenko's theorem and Dual-Glivenko's theorem, respectively [**?**]. □

With this in hand, we have the following result.

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[7]I do not consider implication and co-implication here for the sake of brevity.

Theorem 32. *The combined inference structure $\langle S, \vdash_{LJ}, \dashv_{LDJ} \rangle$, interpreted as a proof and refutation structure, is exactly equivalent by theorems and counter-theorems to the multiple-conclusion inference structure $\langle S, \vdash_{LK} \rangle$.*

Proof. Obvious by Theorem 30. □

Even with these brief comments, we are now in a position to see that there is a significant benefit in splitting apart the structure defined by LK. First, we have a clear account of the conditions under which coherent assertion and denial is derivative of an underlying structure of proofs and refutations. Restall's account of the incoherence of certain assertions and denials can be understood in terms of the conditions upon proofs and refutations in intuitionistic and co-intuitionistic inference structures, respectively.

In addition, because of the way in which we have developed the approach to inferentialist semantics, there is a fairly natural interpretation of the combined inference structure in terms of a Nelson-style model, which reflects something of the constructive nature of the resultant, nonetheless classical, structure. It is natural to construct a Nelson-style structure, for the reason that it allows us to be perspicuous regarding the equal weighting given to proofs and refutations, assertions and denials, truth and falsity. In essence, the construction follows exactly that of the abstract account given above, bearing in mind the constraint given there that, at any stage of the construction of bipartitions, for some α under consideration at that stage, it is either the case that $\emptyset \vdash_{LJ} \alpha$, or $\alpha \dashv_{LDJ} \emptyset$.

Definition 33. Define the semantic structure determined by $\langle S, \vdash_{LJ}, \dashv_{LDJ} \rangle$ as $\mathbb{V} = \langle S, \leq, V^+, V^- \rangle$. In relation to the abstract account given above, we are now explicitly thinking of $\mathbb{V}$ as the collection of "stages" of the process of construction of bipartitions, ranging over the domain of formulas S. V^+, V^- indicate the collection of positive and negative characteristic valuations, respectively (these correspond precisely to Γ^+ and Δ^- above). We will also use $\leq$ to indicate that the structure forms a poset on S, which is obvious from the above proofs. Then, let $\Vdash^+$ and $\Vdash^-$ denote satisfaction and refutation, respectively, and s_i indicate a stage of the construction where $i, 1 \leq i \leq n$.

Definition 34. By the above interpretation of the refutation structure defined by LDJ, and the corresponding structure of proofs for the structure defined by LJ, we can "read off" the following semantic conditions on $\mathbb{V}$:

(1) $s_i \Vdash^+ \alpha$ iff $\alpha \in (V^+, s_i)$
(2) $s_i \Vdash^- \alpha$ iff $\alpha \in (V^-, s_i)$
(3) $s_i \Vdash^+ (\alpha \wedge \beta)$ iff $\alpha \in (V^+, s_i)$ and $\beta \in (V^+, s_i)$
(4) $s_i \Vdash^- (\alpha \wedge \beta)$ iff $\alpha \in (V^-, s_i)$ or $\beta \in (V^-, s_i)$

(5) $s_i \Vdash^+ (\alpha \vee \beta)$ iff $\alpha \in (V^+, s_i)$ or $\beta \in (V^+, s_i)$
(6) $s_i \Vdash^- (\alpha \vee \beta)$ iff $\alpha \in (V^-, s_i)$ and $\beta \in (V^-, s_i)$
(7) $s_i \Vdash^+ (\neg\alpha)$ iff $\alpha \in (V^-, s_i)$
(8) $s_i \Vdash^- (\neg\alpha)$ iff $\alpha \in (V^+, s_i)$

The conditions are straightforward. For example, (1) simply indicates that α is satisfied at stage s_i iff α belongs to the set of assertible formulas at that stage. The negation conditions are natural, reflecting the idea that negation "swaps" a formula between proofs and refutations.

Definition 35. At any stage s_i, we have $V^+(\alpha) = \{\alpha \in S : s_i \Vdash^+ \alpha\}$; $V^-(\alpha) = \{\alpha \in S : s_i \Vdash^- \alpha\}$. We define the relationship between stages in terms of $\leq$, where $s_1 \leq s_2$ iff $s_2 \Vdash^+ \alpha, \forall\alpha \in (V^+, s_1)$ and $s_2 \Vdash^- \alpha, \forall\alpha \in (V^-, s_1)$.

Theorem 36. *Then, $\leq$ is a partial order that is reflexive, monotonic and transitive.*

Proof. Reflexivity is obvious, since $s_i \leq s_i$ for every s_i. For transitivity, assume that $s_1 \leq s_2$ and $s_2 \leq s_3$. Then, $\forall\alpha \in (V^+, s_1), s_2 \Vdash^+ \alpha$ and $\forall\alpha \in (V^-, s_1), s_2 \Vdash^- \alpha$, and $\forall\alpha \in (V^+, s_2), s_3 \Vdash^+ \alpha$ and $\forall\alpha \in (V^-, s_2), s_3 \Vdash^- \alpha$. Consider the case where $\alpha \in (V^+, s_1)$, so $s_2 \Vdash^+ \alpha$, and $s_2 \cup s_3 \Vdash^+ \alpha$ (by $\mathbb{M}$). Applying $\mathbb{T}$ to each side yields $s_3 \Vdash^+ \alpha$. The proof for $\alpha \in (V^-, s_1)$ is analagous. For monotonicity, assume that $\alpha \in (V^+, s_1)$, and $s_1 \leq s_2$, so $s_2 \Vdash^+ \alpha$, and $\alpha \in (V^+, s_2)$. Assume $\alpha \in (V^-, s_1)$, then $s_2 \Vdash^- \alpha$, and $\alpha \in (V^-, s_2)$. □

This property of monotonicity is important since it highlights that the construction is constructible for both V^+ and V^-:

If $s_1 \leq s_2$ then $\forall\alpha \in (V^+, s_1), \alpha \in (V^+, s_2)$; and $\forall\alpha \in (V^-, s_1), \alpha \in (V^-, s_2)$.

This ensures that, whenever a formula is given a valuation at a stage, that valuation remains at every stage upstream, so, if $s_1 \Vdash^+ \alpha$, then $s_2 \Vdash^+ \alpha$; if $s_1 \Vdash^- \alpha$, then $s_2 \Vdash^- \alpha$.

Resultantly, we have a classical structure which is, nonetheless, constructive (over both truth and falsity). For example, if we interpret the above in terms of the set of stages of reasoning over time, in which each stage represents the addition of information, then, the addition of that information will not alter the underlying set of information. The proofs given above show that this can be extended to a bipartition over the entire domain S. This difference is subtle, but nonetheless provides us with a far greater grasp on the internal machinery of the inferential semantics.

4.1. **Further work.** The previous section, in particular, is suggestive of further work that will be pursued in this vein. For example, throughout, we have made two substantive assumptions regarding the combined inference structure which allows that structure to be equivalent to a multiple-conclusion structure. The first was that the set of formulas are allowed to interact freely in the structure. Further work might pursue structures in which the two are restricted in their interaction, perhaps by techniques such as polarization. The second was that the two structures are coherent throughout. In further work, I intend to develop structures which are paracoherent, in the sense that they allow for a limited form of incoherence in specific cases. It is likely that these two areas should be developed in tandem.